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On the indicatrix of orbits of 1-parameter subgroups in a homogeneous space

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§ 1. Preliminaries

In the following $H, K \subset G$ denote Lie groups, $\mathfrak{g}, \mathfrak{h}, \mathfrak{k}$ the corresponding Lie algebras, which can be identified with the tangent spaces T_eG, T_eH, T_eK at the unity $e \in G, H, K$, respectively.

Let be $L(M)$ the bundle of linear frames on the manifold M and $p\colon L(M) \to M$ the natural projection in this bundle.

The isotropy group H of the homogeneous space $M = G/H$ leaves the origin $o \in M$ of the space $M = G/H$ fixed. Hence the differential z_{*o} of the map $z\colon M \to M$ $(z \in H)$ is a linear transformation on the tangent space T_oM. This representation $z \mapsto z_{*o}$ $(z \in H)$ of the isotropy group on the tangent space T_oM is called the linear isotropy group. The action $\alpha\colon G \times M \to M$ of the group G on M induces an action $\tilde{\alpha}\colon G \times L(M) \to L(M)$ of the group G on the linear frame bundle $L(M)$. It is clear that the action $\tilde{\alpha}$ is effective if and only if the linear representation of the isotropy group is faithful, i.e. the map $z \mapsto z_{*o}$ $(z \in H)$ is one-to-one.

It is well-known that the faithfulness of the linear representation of the isotropy group is a necessary condition for the existence of invariant connections in a homogeneous space. Therefore in the following this condition will be supposed.

Let be given a frame $u_0 \in L_oM$ at the point $o \in M$. The action $\tilde{\alpha}$ of G on $L(M)$ yields an embedding of G in $L(M)$ so that to the unity $e \in G$ corresponds the frame u_0. In the following we use this embedding and we will regard the principal bundle $\{G, \pi, G/H\}$ as a subbundle of $\{L(M), p, M\}$.

We recall Wang's theorem on invariant connections, cf. [2], 186—190.

Let be $M = G/H$ a homogeneous space. There exists a one-to-one correspondence between the set of G-invariant connections in $L(M)$ and the set of linear maps $\Lambda\colon \mathfrak{g} \to \mathfrak{gl}(n)$ satisfying the conditions

(i) $\Lambda(X) = \lambda(X)$ if $X \in \mathfrak{h}$,

(ii) $\Lambda([Z, X]) = [\lambda(Z), \Lambda(X)]$ if $Z \in \mathfrak{h}, X \in \mathfrak{g}$,

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where λ denotes the homomorphism of the Lie algebras $\mathfrak{h}\to\mathfrak{gl}(n)$ induced by the linear representation of the isotropy group.

Let φ denote a G-invariant connection form on $L(M)$, than the corresponding linear map $\Lambda\colon \mathfrak{g}\to\mathfrak{gl}(n)$ satisfies

$$\Lambda(X)=\varphi(\hat{X}) \quad \text{if} \quad X\in\mathfrak{g},$$

where $\hat{X}$ denotes the vector field on $L(M)$, defined by the tangent vectors of orbits in $L(M)$ of the one-parameter subgroup $\exp tX\subset G$.

Let $\mathfrak{m}$ denote a complementary subspace to the subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ that is

$$\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}.$$

Let be given a leftinvariant coframe $\{\omega^1, \dots, \omega^n, \omega^{n+1}, \dots, \omega^{n+k}\}$ on the group G such that the equations $\omega^1=\dots=\omega^n=0$ define the subalgebra $\mathfrak{h}$ and the equations $\omega^{n+1}=\dots=\omega^{n+k}=0$ define the subspace $\mathfrak{m}$. In the following the indices have the values: $a, b, c=1, \dots, n$; $\alpha, \beta, \gamma=n+1, \dots, n+k$, where $n=\dim M$ and $n+k=\dim G$. The structure equations of the group G have the form

$$d\omega^a=-\sum_{\beta,c} c^a_{\beta c}\omega^\beta\wedge\omega^c-\frac{1}{2}\sum_{b,c} c^a_{bc}\omega^b\wedge\omega^c,$$

$$d\omega^\alpha=-\frac{1}{2}\sum_{\beta,\gamma} c^\alpha_{\beta\gamma}\omega^\beta\wedge\omega^\gamma-\sum_{\beta,c} c^\alpha_{\beta c}\omega^\beta\wedge\omega^c-\frac{1}{2}\sum_{b,c} c^\alpha_{bc}\omega^b\wedge\omega^c.$$

The connection form φ can be expressed by

$$\varphi(\hat{X})=\sum_{a,c}\varphi^a_c(\hat{X})E^c_a=\sum_{a,c}\Big(\sum_\beta c^a_{\beta c}\omega^\beta(X)+\frac{1}{2}\sum_b c^a_{bc}\omega^b(X)+\frac{1}{2}\sum_b l^a_{bc}\omega^b(X)\Big)E^c_a,$$

where l^a_{bc} are constant and $\{E^c_a\}$ denotes the canonical basis of the linear Lie algebra $\mathfrak{gl}(n)$.

§ 2. The indicatrix of orbits of 1-parameter subgroups

Let be M a differentiable manifold and suppose that there is linear connection on M. Let $y(t)$ be given a differentiable curve in M. The operator of the parallel translation along the curve $y(t)$ will be denoted by $\tau_{t,t_0}\colon T_{y(t)}M\to T_{y(t_0)}M$.

The indicatrix of the curve $y(t)$ at the point $y(t_0)$ is the curve $Y(t)$ in the tangent space $T_{y(t_0)}M$, defined by the parallel translation of the tangent vector $\dot{y}(t)$ of the curve to the point $y(t_0)$:

$$Y(t)=\tau_{t,t_0}\dot{y}(t).$$

Theorem 1. *Let $M=G/H$ be a homogeneous space, and let a G-invariant connection on M be given by a map $\Lambda\colon \mathfrak{g}\to\mathfrak{gl}(n)$, according to Wang's theorem. The indicatrix of the orbit $y(t)=\alpha(\exp tX, o)$ at the origin $o\in M$ $(X\in\mathfrak{g})$ is the curve*

$$Y(t)=\varkappa^{-1}(\exp t\Lambda(X))\varkappa Y_0, \quad \textit{where} \quad \varkappa\colon T_0M\to\mathbf{R}^n \quad \textit{is the coordinate map}$$

defined by the frame u and $Y_0=\pi_*(X)\in T_oM$ is the tangent vector to the curve $y(t)$ at the initial point o.

Proof. Since we regard the group G as a submanifold of $L(M)$, the 1-parameter subgroup $x(t)=\exp tX$ $(X\in \mathfrak{g})$ is a curve in $L(M)$ with tangent vectors $\hat{X}(t)\in T_{x(t)}L(M)$. The equations of $x(t)$ in $G\subset L(M)$ are

$$\frac{d}{dt}\omega^a(\hat{X}(t))=0\ \ (a=1,\dots,n),\quad \frac{d}{dt}\omega^\alpha(\hat{X}(t))=0\ \ (\alpha=n+1,\dots,n+k),$$

with respect to the given G-left invariant coframe $\{\omega^1,\dots,\omega^{n+k}\}$. Hence the equations of the orbit $y(t)=\alpha(\exp tX, o)=p\cdot x(t)$ are

$$\frac{d}{dt}\omega^a(\hat{X}(t))=0\quad (a=1,\dots,n).$$

On the other hand, using the following lemma, the components of the covariant derivative $\nabla_t\dot{y}=\nabla_{\frac{\partial}{\partial t}}\dot{y}$ of the tangent vector $\dot{y}(t)$ of the orbit $y(t)$ can be expressed as

$$\omega^a(\nabla_t\dot{y})=\frac{d}{dt}\omega^a(\hat{X})+\sum_c\varphi^a_c(\hat{X})\omega^c(X).$$

Lemma. *Let M be a manifold equipped with a connection form φ on $L(M)$. Let $y(t)$ be a curve in M, $X(t)$ a vector field along $y(t)$. The components $\omega^1,\dots,\omega^n$ of the R^n-valued canonical form ω on the covariant derivative vector $\nabla_tX=\nabla_{\frac{\partial}{\partial t}}X$ along the curve $y(t)$ satisfy*

$$\omega^a(\nabla_tX)=\frac{d}{dt}\omega^a(\hat{X})+\sum_c\varphi^a_c(\hat{y})\omega^c(\hat{X})$$

where $\hat{y}$ and $\hat{X}$ denote the horizontal lifts of the vectors $\dot{y}$ and X, and φ^a_c are the components of connection form φ.

This lemma is a version of Theorem 11 in § 6.4 [1]. $\Lambda(X)=\varphi(\hat{X})$ and $\frac{d}{dt}\omega^a(X)=0$, we get $\nabla_t\dot{y}=\varkappa^{-1}\Lambda(X)\varkappa\dot{y}$, where $\varkappa\colon T_0M\to\mathbf{R}^n$ is the coordinate map defined by the chosen frame field, or equivalently, we get the equation of the indicatrix $Y(t)$ of $y(t)$ in the form

$$\frac{d}{dt}Y(t)=\varkappa^{-1}\Lambda(X)\varkappa Y(t).$$

It is well-known that the solution of this ordinary differential equation with constant coefficients is

$$Y(t)=\varkappa^{-1}(\exp t\Lambda(X))\varkappa Y_0,$$

where $Y_0=Y(0)=\pi_*X$. The theorem is proved.

Corollary. *The k-th covariant derivative $\nabla^{(k)}_t\dot{y}$ of tangents of the orbit $y(t)=\alpha(\exp tX, o)$ at the initial point $o\in M$ is $(\Lambda(X))^kY_0$, where $Y_0=\pi_*Y$.*

§ 3. The indicatrix of orbits in a reductive space

If there is given a reductive complement $\mathfrak{m}\subset\mathfrak{g}$ to the subalgebra $\mathfrak{h}$ in the Lie algebra $\mathfrak{g}$, characterized by

$$\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m} \quad\text{and}\quad [\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m},$$

then it is clear that the map $\Lambda\colon \mathfrak{g}\to\mathfrak{gl}(n)$ defined by

$$\Lambda(X)=\lambda(X)\quad\text{if}\quad X\in\mathfrak{h},\quad \Lambda(X)=0\quad\text{if}\quad X\in\mathfrak{m}$$

satisfies the assumptions of Wang's theorem. The corresponding G-invariant connection is called the canonical connection of the reductive space $\{M=G/H,\mathfrak{m}\}$. From Theorem 1 it follows immediately:

Theorem 2. *Let $\{M=G/H,\mathfrak{m}\}$ be a reductive homogeneous space. The curve $y(t)$ in M is the orbit of a 1-parameter subgroup of G if and only if its indicatrix with respect to the canonical connection is an orbit of a 1-parameter subgroup of linear isotropy group. In detail, the indicatrix of the orbit $\alpha(\exp tX,o)$ at the origin $o\in M$ is the curve $Y(t)=(\exp t\,\mathrm{ad}\,Z)Y_0$, where $Z=X_{\mathfrak{h}}$ and $Y_0=X_{\mathfrak{m}}$ are the components of the vector X in the subspaces $\mathfrak{h}$ and $\mathfrak{m}$, respectively, and the tangent space T_oM is identified with the reductive complement $\mathfrak{m}$.*

Proof. From the property $[\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m}$ of the reductive complement $\mathfrak{m}$ follows that the homomorphism $\lambda\colon\mathfrak{h}\to\mathfrak{gl}(n)$ induced by the linear representation of isotropy group has the form: $\lambda(Z)=\mathrm{ad}\,Z\colon\ \mathfrak{m}\to\mathfrak{m}$ $(Z\in\mathfrak{h})$. The theorem is proved.

Corollary. *The k-th covariant derivative $\nabla_t^{(k)}\dot y$ of the tangents of the orbit $y(t)=\alpha(\exp tX,o)$ at the initial point $o\in M$ is* $(\mathrm{ad}\,Z)^k Y_0$.

§ 4. Geodesics in a fibering of reductive space

Let $\{M=G/H,\mathfrak{m}\}$ be a reductive homogeneous space. Let be given a subgroup $K\subset H$ and a reductive complementum $\mathfrak{f}$ on the homogeneous space $F=H/K$. The homogeneous space $N=G/K$ has a structure of a fibre bundle $\{N,\pi,M,F\}$, where N, M and F are the total, basic and the fiber type manifolds, respectively. We have the decompositions of Lie algebras

$$\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m},\quad \mathfrak{h}=\mathfrak{k}\oplus\mathfrak{f},\quad \mathfrak{g}=\mathfrak{k}\oplus\mathfrak{f}\oplus\mathfrak{m}$$

satisfying

$$[\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m},\quad [\mathfrak{k},\mathfrak{f}]\subset\mathfrak{f},\quad [\mathfrak{k},\mathfrak{f}\oplus\mathfrak{w}]\subset\mathfrak{f}\oplus\mathfrak{m}.$$

It is clear that $\mathfrak{f}\oplus\mathfrak{m}$ is a reductive complement on the homogeneous space $N=G/K$.

We investigate the projection to M of the geodesics in the homogeneous space $N=G/K$ with respect to the canonical connection corresponding to the reductive complement $\mathfrak{f}\oplus\mathfrak{m}$.

Theorem 3. *The curve $y(t)$ in $M=G/H$ through the origin $o\in M$ is a projection of a geodesic in $N=G/K$ $(K\subset H)$ with respect to the canonical connection if and only if its indicatrix at the origin $o\in M$ is an orbit of a 1-parameter subgroup* $\exp t \operatorname{ad} Z$ *of the linear isotropy group, where* $Z\in\mathfrak{k}$.

(Here and in the following $\operatorname{ad} Z\colon \mathfrak{g}\to\mathfrak{g}$ denotes the operator $X\to[Z, X]$ on $\mathfrak{g}$. Since $[\mathfrak{h}, \mathfrak{m}]\subset\mathfrak{m}$, this operator can be restricted to the subspace $\mathfrak{m}\subset\mathfrak{g}$; this restriction is denoted by the same way.)

Proof. Since $N=G/K$ is a reductive homogeneous space equipped with canonical connection, the geodesics in N are the orbits of 1-parameter subgroups $\exp tX$ of the group G, where $X\in\mathfrak{k}\oplus\mathfrak{m}$. From Theorem 2, it follows that the indicatrix of the orbit of subgroup $\exp tX$ at the point $o\in M$ is the curve $Y(t)=(\exp t \operatorname{ad} Z)Y$, where $Z=X_{\mathfrak{h}}$ and $Y=X_{\mathfrak{m}}$. From $X\in\mathfrak{k}\oplus\mathfrak{m}$ follows that $Z=X_{\mathfrak{h}}\in\mathfrak{k}$.

On the other hand, if $Y\in\mathfrak{m}(=T_oM)$, $Z\in\mathfrak{k}$, then it is clear that $Y(t)=(\exp t \operatorname{ad} Z)Y$ is the indicatrix of the orbit of the subgroup $\exp t(Y+Z)$. But we know that the orbit of a 1-parameter subgroup $\exp t(Y+Z)$ in the space $N=G/K$ is geodesic. The theorem is proved.

§ 5. Geodesics in the tangent sphere bundle of a 2-transitive Riemannian homogeneous space

We apply our results to the characterization of the projections of geodesics of the tangent sphere bundle of a 2-transitive Riemannian homogeneous space with respect to the Sasaki metric. We get a generalization of a result ([5], [4], [3]) asserting that the projection of a geodesic of the tangent sphere bundle of a space of constant curvature is a helix.

Let be $M=G/H$ a 2-transitive Riemannian homogeneous space, that is the group G is supposed to act transitively on the tangent sphere bundle N of the manifold M. It is well-known that from the 2-transitivity of the isometry group G of M follows that M is symmetric space (cf. [6], 289). On a Riemannian symmetric space $M=G/H$ there is a natural reductive complement $\mathfrak{m}\subset\mathfrak{g}$ whose canonical connection has the same geodesics as the Riemannian connection of the symmetric space M [6].

From the 2-transitivity of G on $M=G/H$ it follows that there exists a subgroup $K\subset H$ such that the tangent sphere bundle N can be written in the form $N=G/K$. The isotropy group H is isomorphic to a subgroup of the orthogonal group $O(n)$, and hence we have an invariant metric on H. This metric induces on the homogeneous space $F=H/K$ a naturally reductive Riemannian metric, which defines on F the geometry of n-sphere. Let $\mathfrak{m}$ and $\mathfrak{k}$ denote the reductive complements on M

and F, respectively, i.e. we have $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$, $\mathfrak{h}=\mathfrak{k}\oplus\mathfrak{f}$. Now we can apply Theorem 3 to this case.

Theorem 4. *Let $M=G/H$ be a 2-transitive Riemannian homogeneous space. The curve $y(t)$ in M is a projection of a geodesic in the tangent sphere bundle if and only if $y(t)$ is a 3-dimensional helix (i.e. the first two curvatures $\varkappa_1$, $\varkappa_2$ are arbitrary constants, and the others zero: $\varkappa_3=\ldots=\varkappa_{n-1}=0$).*

Proof. From Theorem 3 we know that $y(t)$ is a projection of a geodesic in N if and only if its indicatrix has the form $\exp(t \operatorname{ad} Z)Y$, where $Y\in\mathfrak{m}$, $Z\in\mathfrak{f}\subset\mathfrak{h}$.

After identifying an orthogonal frame at $o\in M$ with the identity of H the adjoint representation maps the group H isomorphically on a subgroup of the orthogonal group $O(n)$ acting on the unit $(n-1)$-sphere of the tangent space T_oM $(=\mathfrak{m})$. In the following we identify the group H with the subgroup of $O(n)$ by this isomorphism. The reductive complement $\mathfrak{f}$ of the subalgebra $\mathfrak{k}$ in $\mathfrak{h}$ corresponds to the tangent space at the initial point of the $(n-1)$-sphere $F=H/K$. Since the reductive complement $\mathfrak{f}$ on $F=H/K$ is identified with the reductive complement on the $(n-1)$-sphere $S^{n-1}=O(n)/O(n-1)$, the 1-parameter subgroup $\exp(t \operatorname{ad} Z)$ $(Z\in\mathfrak{f})$ of $O(n)$ is a 1-parameter rotation group around the $(n-2)$-plane in T_oM, orthogonal to the 2-plane of the geodesic great circle which is the orbit of $\exp(t \operatorname{ad} Z)$ in $S^{n-1}=F$ through the initial point. It follows that the curve $Y(t)=\exp(t \operatorname{ad} Z)Y$ $(Y\in\mathfrak{m}, Z\in\mathfrak{f})$ is a circle. The indicatrix of a curve $y(t)$ is a circle if and only if it is a 3-dimensional helix. Theorem 4 is proved.

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